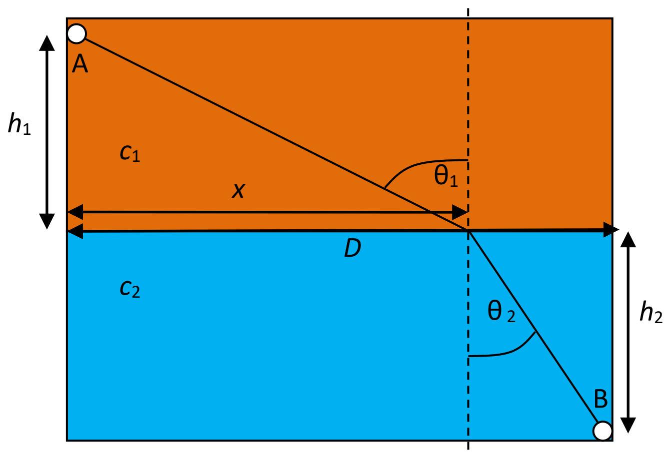
Teacher notes
Topic C

Refraction and the problem of least time

You are on sand at position A. A position B is in water. You want to get from A to B by running part of the way on sand and then swimming the rest of the way in water. Your speed on sand is c_1 and that in water is a lower speed c_2. This is related to refraction. The path you follow is the path followed by a ray of light from A to B when there is a change of medium, i.e. refraction.

You must decide where you will enter the water in order to get from A to B in the least possible amount of time. In other words you must find the optimal distance *x*.



Time taken on sand: $\dfrac{\sqrt{h_1^2+x^2}}{c_1}$.

Time taken in water: $\dfrac{\sqrt{h_2^2+(D-x)^2}}{c_2}$.

Total time is $T=\dfrac{\sqrt{h_1^2+x^2}}{c_1}+\dfrac{\sqrt{h_2^2+(D-x)^2}}{c_2}$. We want to find *x* such that this time is as small as possible: so we must differentiate *T* and set the derivative to zero.

$$\frac{dT}{dx}=\frac{x}{c_1\sqrt{h_1^2+x^2}}-\frac{D-x}{c_2\sqrt{h_2^2+(D-x)^2}}=0$$

which gives

$$\frac{x}{c_1\sqrt{h_1^2+x^2}}=\frac{D-x}{c_2\sqrt{h_2^2+(D-x)^2}}.$$

But

$\frac{x}{\sqrt{h_1^2+x^2}}=\sin\theta_1$ and $\frac{D-x}{\sqrt{h_2^2+(D-x)^2}}=\sin\theta_2$ so we get

$$\frac{\sin\theta_1}{c_1}=\frac{\sin\theta_2}{c_2}$$

i.e. Snell's law.

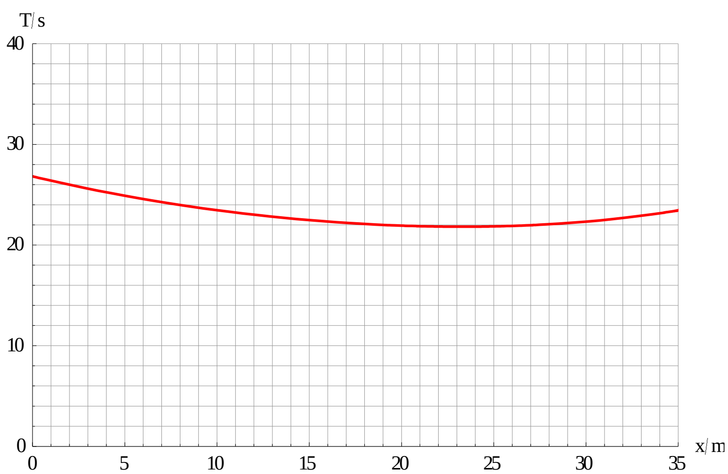
It appears that light gets from A to B in the shortest possible time!

You can avoid calculus by putting in numbers and using the GDC.

For example, let, $h_1=h_2=20$ m, $D=35$ m and $c_1=3\text{ m s}^{-1}$, $c_2=2\text{ m s}^{-1}$, then

$$T=\frac{\sqrt{400+x^2}}{3}+\frac{\sqrt{400+(35-x)^2}}{2}$$

Plotting the time T we get:



The minimum time is obtained for $x\approx 23$ m. Then $\tan\theta_1=\frac{23}{20}\Rightarrow\theta_1\approx 49^\circ$ and $\tan\theta_2=\frac{35-23}{20}\Rightarrow\theta_2\approx 31^\circ$.

Then

$$\frac{\sin\theta_1}{\sin\theta_2}\approx 1.5=\frac{c_1}{c_2}$$

The French mathematician Pierre de Fermat elevated this to a principle, the principle of least time, in 1662: light travels from A to B in the least possible time. In ancient Greece, Euclid and Hero of Alexandria had shown that in reflection, a ray of light follows the path of least length and time.

The principle has been extended to the principle of least action in Lagrangian mechanics that has made possible the transition from classical to quantum mechanics, among many other things. You may want to know more about all this as well as to discuss the question of how light "knows" which is the path of least time!